Phases of Partial Matrix Elements and the Parity Rule for Inelastic Scattering*

I. E. McCarthy
Department of Physics, University of California, Davis, California

(Received 6 February 1964)

It is shown that the phases of the partial matrix elements in the distorted-wave Born approximation, which are responsible for effects like forward and backward peaks in angular distributions, are closely related to the elastic scattering phase shifts for the entrance and exit channels and hence largely independent of the particular optical model used to calculate the wave functions. Phase considerations are used to verify the parity rule for even angular momentum transfer, and the conditions under which it is expected to hold are discussed.

1. INTRODUCTION

The parity rule for inelastic scattering at small angles was discovered in numerical calculations by Glendenning[1] and proved by Kromminga and McCarthy.[2] In its most restricted form the rule states that if the Q value is zero, if the entrance and exit channel optical-model potentials are identical, if there is no space-exchange term in the interaction, and if the parity of the nuclear state is changed, the distorted-wave Born approximation gives zero for the differential cross section for forward scattering.

When the first three conditions hold approximately, the forward cross section is expected to be small, or at least to decrease as the scattering angle θ approaches zero. Inelastic proton scattering experiments[3] in which the parity is known to change have confirmed that at least the cross section decreases towards small angles. Cross sections have been observed down to about 15°. The decrease generally starts at about 30 or 40°. The effect is seen at larger angles with lighter incident particles because the momentum transfer is less and the differential cross section fluctuates less rapidly.

It has also been observed that the differential cross section generally rises as the scattering angle approaches zero if the parity of the nucleus does not change and the Q value is not too large. If this is a general rule, it makes the parity rule a more effective tool for nuclear spectroscopy since it would then be possible to say that a reaction with a decreasing forward cross section is definitely one which changes the parity. At present it is only possible to say that a reaction with an increasing forward cross section does not change the parity.

The parity-changing (odd L) part of the rule is an exact selection rule involving the fact that, under the above adiabatic conditions,

$$I_{L,l'l}{}^{0}=(-1)^{L}I^{0}{}_{L,ll'}\,, \tag{1}$$

where $I_{L,ll'}{}^{0}$ is the partial matrix element in the distorted-wave Born approximation for the lth partial wave in the entrance channel and the l'th partial wave in the exit channel. The diagonal terms $I_{L,ll}{}^{0}$ do not contribute when the parity is changed because of the selection rule $l+l'+L$ even. L is the angular momentum transfer. Only the $M=0$ components contribute for $\theta=0$.

It was shown in Ref. 2 by means of a simple approximation for the entrance and exit channel optical-model wave functions, which involved focusing, that the focusing is responsible for large forward cross sections for even L.

In this note the conditions for the second part of the parity rule, namely, that forward cross sections are increasing towards zero scattering angle for even L, will be established. Unlike that for the first part of the rule, which is a clear-cut selection rule, the argument for the second part of the rule is approximate. It is slightly similar to the argument[4] for the Blair phase rule[5] for surface inelastic scattering.

In Sec. 2 it will be shown that focusing is a general property of an elastic scattering wave function. It is a necessary consequence of the phase shifts for the partial waves.

In Sec. 3 it will be shown under what circumstances the focusing property is expected to lead to large, or at least increasing, forward cross sections.

2. FOCUSING AS A CONSEQUENCE OF REAL PHASE SHIFTS

The optical-model wave function $\chi^{(+)}(\mathbf{k},\mathbf{r})$ will be written in the following way:

$$\chi^{(+)}(\mathbf{k},\mathbf{r})=\sum_{l=0}^{\infty}\chi_{l}(\rho)P_{l}(\cos\theta)\,, \tag{2}$$

* Supported by the U. S. Atomic Energy Commission.

[1] N. K. Glendenning, Phys. Rev. **114**, 1297 (1959).

[2] A. J. Kromminga and I. E. McCarthy, Phys. Rev. Letters **6**, 62 (1961).

[3] For example, N. Hintz and T. Stovall, in University of Minnesota Linear Accelerator Laboratory Annual Report, 1962 (unpublished).

[4] See, for example, N. Austern, in *Selected Topics in Nuclear Theory* (International Atomic Energy Agency, Vienna, 1963), p. 39.

[5] J. S. Blair, in *Proceedings of the International Conference on Nuclear Structure, Kingston*, edited by D. A. Bromley and E. W. Vogt (The University of Toronto Press, Toronto, 1960).

where

$$\rho = kr, \tag{3}$$

$$x_l(\rho) = i^l(2l+1)f_l(\rho)\exp(i\sigma_l). \tag{4}$$

σ_l is the Coulomb phase shift.

Normally, the form factor for the optical-model potential has zero derivative at the origin. The potential is flat in the middle. Hence, the solution to the radial Schrödinger equation at the origin is

$$f_l(\rho) = j_l(\rho')\exp(i\phi_l), \tag{5}$$

where

$$\rho' = k'r, \tag{6}$$

$$k' = \{(2m/\hbar^2)(E-V-iW)\}^{1/2}, \tag{7}$$

that is, the wave function at the origin is a spherical Bessel function in the flat internal potential except for a phase ϕ_l. The phase ϕ_l is obtained by matching the internal solution to the asymptotic solution at a radius ρ_0.

$$f_l(\rho) \sim F_l(\rho) + C_l[G_l(\rho) + iF_l(\rho)]. \tag{8}$$

F_l and G_l are the Coulomb functions, regular and irregular at the origin, respectively. C_l is given by

$$e^{2i\delta_l} = 2iC_l + 1, \tag{9}$$

where δ_l is the phase shift.

We will prove that $\phi_l = \delta_l$.

The matching condition for C_l is given by

$$C_l = -B/(A+iB), \tag{10}$$

where A and B are the following Wronskians, calculated at the matching radius ρ_0.

$$A = W(G_l, f_l),$$
$$B = W(F_l, f_l). \tag{11}$$

From Eqs. (9) and (10), we obtain

$$\tan\delta_l = -B/A. \tag{12}$$

The radial wave function is defined near the origin by $j_l(\rho')$ except for a phase factor $\exp(i\phi_l)$. The normalization is given by matching internal and external solutions at ρ_0.

$$\exp(i\phi_l) = \{F_l(\rho_0) + C_l[G_l(\rho_0) + iF_l(\rho_0)]\}/f_l(\rho_0). \tag{13}$$

Using Eqs. (10), (11), and (13), we obtain

$$\tan\phi_l = -B/A. \tag{14}$$

Thus, it has been shown that the phase ϕ_l is the phase shift δ_l. It must be noted that when the potential is complex, ϕ_l is complex. $j_l(\rho')$ is also a complex number, so that the magnitude and phase of f_l at the origin are given by $j_l(\rho')$ and $\phi_l(\rho)$, respectively, only for real potentials. The magnitude and phase of f_l are close to the real parts of $j_l(\rho')$ and $\phi_l(\rho)$ if W is small compared to $E+V$, which is usually the case.

This is a particular case of the well-known fact for potential scattering[6] that

$$\psi_l(k,r) = k^l|f_l(k)|^{-1}g_l(k^2,r)e^{i\delta_l}, \tag{15}$$

where $f_l^*(k) = f_l(-k)$ and $f_l(k)$ are Jost functions. $f_l(-k)$ is the Fredholm denominator of the integral equation for the lth partial wave. $g_l(k^2,r)$ is real.

It has been shown in a previous publication[7] that the differences between the phases of successive partial waves near the nuclear surface are responsible for focusing. The phase of the lth partial wave near the first peak (i.e., for $\rho \cong l$), which gives the major contribution to the expansion (2) for $x^{(+)}$, is nearly the same as it is at the origin.[7] Hence, for the purpose of understanding the interference of partial waves, we may consider the phase of the lth partial wave to be Re δ_l.

The argument for the focus may be summarized as follows. The differences between the phase shifts for partial waves of low l are small. The differences are large for surface partial waves, and again small for large l. At $\theta = 0$, the large differences in phase for successive surface partial waves cause constructive interference for values of ρ near the surface.

It has thus been shown that focusing in an elastic scattering wave function is independent of any particular model and is a property of the phase shifts.

In Ref. 7 it has also been shown how focusing is responsible for qualitative features of angular distributions such as backward peaks.

3. FORWARD INELASTIC SCATTERING IN THE ADIABATIC LIMIT

Using the distorted-wave Born approximation we will consider inelastic scattering in the approximation that the radial wave functions $f_l(\rho)$ and $f_{l'}(\rho)$ for the initial and final states are identical.

Taking the case of spinless incident particles for simplicity, the differential cross section may be written in the usual notation as

$$(d\sigma/d\Omega)(\theta) = (\mu/2\pi\hbar^2)^2(k'/k)\textstyle\sum_{\rm av}|M_L{}^M(\theta)|^2, \tag{16}$$

where, neglecting space exchange,

$$M_L{}^M(\theta) = \textstyle\sum_{ll'} I_{L,ll'}{}^M Y_{l'}{}^M(\theta,0). \tag{17}$$

The selection rules are as follows: $l+l'+L$ even and l,l', L obey triangle inequalities.

For forward scattering $Y_{l'}{}^M(\theta,0)$ is zero for $M \neq 0$. We may illustrate the argument by considering the relevant partial matrix element for a single-particle interaction.

$$I_{L,ll'}{}^0 = i^{l-l'}\left(\frac{2L+1}{4\pi}\right)^{1/2} e^{i(\sigma_l+\sigma_{l'})} R_{ll'L} 4\pi(2l'+1)(C_{lLl'}{}^{000})^2 \times\langle j'l'm'|Y_L{}^0|jlm\rangle, \tag{18}$$

[6] R. G. Newton, J. Math. Phys. **1**, 319 (1960).

[7] I. E. McCarthy, Phys. Rev. **128**, 1237 (1962).

where

$$R_{ll'L}=\int r^2 dr f_l(k,r) f_{l'}(k,r) H_L(r)\,,$$

$$H_L(r)=\int r'^2 dr' \phi^*{}_{l'}(r') v_L(r,r') \phi_l(r')\,. \tag{19}$$

The orbital, total, and projection angular momentum quantum numbers for the initial nuclear state are l, j, m, respectively, the radial wave function for the initial state is $\phi_l(r')$, the primed quantum numbers indicate the final state and v_L is the coefficient in the expansion of the two-body potential in spherical harmonics.

Using the selection rule that $l+l'+L$ is even, it is clear from Eqs. (18) and (19) that

$$I_{L,l'l}{}^0=(-1)^L I_{L,ll'}{}^0\,. \tag{20}$$

For odd values of L, the (l,l') terms in the sum (16) exactly cancel the (l',l) terms. This is the first part of the parity rule.

The second part of the rule is seen by considering the plane-wave case where we put

$$x^{(+)}(\mathbf{k},\mathbf{r})=\exp(i\mathbf{k}\cdot\mathbf{r})\,. \tag{21}$$

This gives

$$f_l(\rho)=j_l(\rho)\,. \tag{22}$$

It is well known that for $L\neq 0$, $M_L{}^0=0$ in this case. Equation (17) may be written as

$$M_L{}^0(0)=\int r^2 dr H_L(r) \times\{\sum_{ll'} i^{l-l'} 4\pi(2l'+1)(C_{lLl'}{}^{000})^2 j_l(kr) j_{l'}(kr)\} \times\left(\frac{2L+1}{4\pi}\right)^{1/2}\langle j'l'm'|Y_L{}^0|jlm\rangle\,. \tag{23}$$

For $L=0$, we have only the diagonal terms in the sum, which clearly add to give a large cross section.

For $L\neq 0$ and even, we have the diagonal terms in the sum plus terms close to the diagonal for small L. Since $H_L(r)$ is an arbitrary function, the integral is only zero if the sum is zero for all r. Thus the terms in the sum cancel, but in a complicated way that depends on r.

At this stage we consider only the terms which contribute most to the integral, that is the terms for $l\approx\lambda$,[8,9] where λ is the surface value of the angular momentum. The argument is now in the spirit of the argument for the Blair phase rule[4] where an average over l values near λ is considered.

The important terms are

$$I_{\lambda-1,\lambda-1}, I_{\lambda,\lambda}, I_{\lambda+1,\lambda+1}, I_{\lambda-1,\lambda+1}, I_{\lambda,\lambda-2}, I_{\lambda,\lambda+2}\,,$$

and terms with l and l' interchanged. These terms are all real for plane waves and they add up to a small number, since the remaining terms are all small. Consider these terms as complex numbers in an Argand diagram. Table I shows their phases, taking the phase of $I_{\lambda\lambda}$ as zero.

TABLE I. Phases of dominant terms in $M_L{}^M(0)$.

l	l'	Phase: Plane waves	Phase: Distorted waves
$\lambda-1$	$\lambda-1$	0	$2(\delta_{\lambda-1}-\delta_\lambda)$
λ	λ	0	0
$\lambda+1$	$\lambda+1$	0	$2(\delta_{\lambda+1}-\delta_\lambda)$
$\lambda-2$	λ	π	$\pi+\delta_{\lambda-2}-\delta_\lambda$
λ	$\lambda+2$	π	$\pi+\delta_{\lambda+2}-\delta_\lambda$
$\lambda-1$	$\lambda+1$	π	$\pi+\delta_{\lambda-1}+\delta_{\lambda+1}-2\delta_\lambda$

For distorted waves, making the approximation that the phase of f_l where it contributes most to the integral is δ_l, we can see from the last column of Table I how the phase shifts affect the cancellation.

For protons with low ($\sim$20 MeV) energies, the differences between the phase shifts for successive surface partial waves can be about 45°. This causes some of the terms to change direction by large amounts relative to $I_{\lambda\lambda}$. Thus we have constructive interference for distorted waves instead of the destructive interference for plane waves.

For partial waves with low l, the most important contribution to the integral is from small r. Successive partial waves have similar phase shifts, so the inside of the nucleus does not contribute much to the forward cross section. This is not true at low energies where the surface l value, λ, is itself small. In fact it is clear that the partial waves which contribute most to the forward cross section are the ones which are responsible for the focusing effects. The forward cross section, like the focus, is due to constructive interference between successive partial waves whose phases are very different. Since the phases of the partial waves are approximately equal to the phase shifts, it may be said that the forward inelastic scattering is a property of the phase shifts and not of a particular model.

The condition for a large forward cross section is that for $l\approx\lambda$, $\delta_l-\delta_\lambda$ must be large enough for significant constructive interference. This condition is fulfilled for low- and medium-energy nucleons.

For heavier ions the imaginary parts of the phase shifts are larger relative to the real parts, and the approximation that the phase of the partial wave is the real part of the phase shift is not such a good one. Numerical computations of α-particle inelastic scattering[10] show that the parity rule still holds for small enough Q values. In fact the even L cases show increasing cross sections as θ approaches zero provided the

[8] E. Rost, Phys. Rev. **128**, 2708 (1962).
[9] K. A. Amos and I. E. McCarthy, Phys. Rev. **132**, 2261 (1963).
[10] R. H. Bassel, G. R. Satchler, R. M. Drisko, and E. Rost, Phys. Rev. **128**, 2693 (1962).

Q value is less than about one-tenth of the incident energy.

The case $L=0$ is slightly special. Here the phase shifts lessen the constructive interference that occurs for plane waves where only the diagonal terms $I_{L,ll}{}^0$ contribute to the sum. In all cases the forward cross section oscillates in magnitude as the energy increases. There will always be an energy at which it is large.

Having discussed the conditions under which the forward cross section is expected to be large for even L and small for odd L, we must now discuss whether the cross section increases or decreases as θ tends to zero.

In the odd L case it is clear that the cross section is very small for $\cos\theta=1$ because of a cancellation. As $1-\cos\theta$ increases, the (l',l) terms in the sum (17) become different from the (l,l') terms. The difference is responsible for the cross section and it increases as θ increases as long as the dominant spherical harmonics (those for $l'\approx\lambda$) remain approximately in phase.

In the even L case the forward cross section is again due mainly to the addition of terms with $l,l'\approx\lambda$ in the sum (17). These terms become smaller as θ increases. The shape of the angular distribution is something like the average of $P_l(\cos\theta)$ for $l\approx\lambda$.

For larger θ the contributions for $M\neq 0$ are not negligible. This also means that θ must be quite small if the increase or decrease of the cross section is to be a useful indication of the parity change.

For 44 MeV α-particle inelastic scattering with even L, the characteristic increase occurs in numerical calculations[10] only for angles less than 5°. λ in this case is about 10. For proton scattering at 10–20 MeV, λ is about 4 or 5 so the characteristic increase can be seen at larger angles.

4. CONCLUSIONS

It has been shown in Sec. 3 how the phases of the partial waves are responsible for large forward cross sections in the case of even $L\neq 0$. In earlier publications[7,9] it was shown how the phases also can be easily seen to control the backward cross sections. Blair[11] has given a very simple illustration of this fact by showing that just reversing the phase of one contribution to the matrix element from a surface reaction can change the angular distribution from an over-all increase towards forward angles to an over-all increase towards backward angles.

The result to be stressed is that of Sec. 2 which shows that these quite detailed considerations of angular distributions are directly related to the elastic scattering phase shifts for the entrance and exit channels and are not expected to depend on the particular model used to calculate the elastic scattering wave function. A dispersion-relation calculation as suggested by Kaminsky and Orlov[12] in which distortion is introduced only by the phase shifts must presumably lead to qualitatively similar angular distributions.

The important practical result is that the parity rule is now established for even L as well as for odd L.

---

[11] J. S. Blair, in *Direct Interactions and Nuclear Reaction Mechanisms* (Gordon and Breach Publishers, Inc., New York, 1963), p. 1165.

[12] V. A. Kaminsky and Yu. V. Orlov, Nucl. Phys. **43**, 236 (1963); **48**, 375 (1963).